4

Principle of Mathematical Induction

Short Answer Type Questions

Q. 1 Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

Sol. Let the statement $P(n):\ 3n < n!$

For $n = 1$, $3 \times 1 < 1!$ [false]

For $n = 2$, $3 \times 2 < 2! \Rightarrow 6 < 2$ [false]

For $n = 3$, $3 \times 3 < 3! \Rightarrow 9 < 6$ [false]

For $n = 4$, $3 \times 4 < 4! \Rightarrow 12 < 24$ [true]

For $n = 5$, $3 \times 5 < 5! \Rightarrow 15 < 5 \times 4 \times 3 \times 2 \times 1 \Rightarrow 15 < 120$ [true]

Q. 2 Give an example of a statement $P(n)$ which is true for all n. Justify your answer.

Sol. Consider the statement

$$P(n): 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$,
$$1 = \frac{1(1+1)(2 \times 1 + 1)}{6}$$

$$\Rightarrow \quad 1 = \frac{2(3)}{6}$$

$$\Rightarrow \quad 1 = 1$$

For $n = 2$,
$$1 + 2^2 = \frac{2(2+1)(4+1)}{6}$$

$$\Rightarrow \quad 5 = \frac{30}{6} \Rightarrow 5 = 5$$

For $n = 3$,
$$1 + 2^2 + 3^2 = \frac{3(3+1)(7)}{6}$$

$$\Rightarrow \quad 1 + 4 + 9 = \frac{3 \times 4 \times 7}{6}$$

$$\Rightarrow \quad 14 = 14$$

Hence, the given statement is true for all n.

Prove each of the statements in the following questions from by the Principle of Mathematical Induction.

Q. 3 $4^n - 1$ is divisible by 3, for each natural number n.

Thinking Process

In step I put $n=1$, the obtained result should be a divisible by 3. In step II put $n=k$ and take $P(k)$ equal to multiple of 3 with non-zero constant say q. In step III put $n=k+1$, in the statement and solve till it becomes a multiple of 3.

Sol. Let $P(n): 4^n - 1$ is divisible by 3 for each natural number n.

Step **I** Now, we observe that $P(1)$ is true.

$$P(1) = 4^1 - 1 = 3$$

It is clear that 3 is divisible by 3.

Hence, $P(1)$ is true.

Step **II** Assume that, $P(n)$ is true for $n = k$

$P(k): 4^k - 1$ is divisible by 3

$$x 4^k - 1 = 3q$$

Step **III** Now, to prove that $P(k+1)$ is true.

$$P(k+1): 4^{k+1} - 1$$
$$= 4^k \cdot 4 - 1$$
$$= 4^k \cdot 3 + 4^k - 1$$
$$= 3 \cdot 4^k + 3q \qquad [\because 4^k - 1 = 3q]$$
$$= 3(4^k + q)$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all natural number n.

Q. 4 $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Sol. Let $P(n): 2^{3n} - 1$ is divisible by 7

Step **I** We observe that $P(1)$ is true.

$$P(1): 2^{3\times 1} - 1 = 2^3 - 1 = 8 - 1 = 7$$

It is clear that $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$,

$P(k): 2^{3k} - 1$ is divisible by 7.

$$\Rightarrow \quad 2^{3k} - 1 = 7q$$

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): 2^{3(k+1)} - 1$$
$$= 2^{3k} \cdot 2^3 - 1$$
$$= 2^{3k}(7+1) - 1$$
$$= 7 \cdot 2^{3k} + 2^{3k} - 1$$
$$= 7 \cdot 2^{3k} + 7q \qquad \text{[from step II]}$$
$$= 7(2^{3k} + q)$$

Hence, $P(k+1)$: is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all natural number n.

Q. 5 $n^3 - 7n + 3$ **is divisible by 3, for all natural numbers** n.

Sol. Let $P(n): n^3 - 7n + 3$ is divisible by 3, for all natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1) = (1)^3 - 7(1) + 3$$
$$= 1 - 7 + 3$$
$$= -3, \text{ which is divisible by 3.}$$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$\therefore \quad P(k) = k^3 - 7k + 3 = 3q$$

Step **III** To prove $P(k+1)$ is true

$$P(k+1): (k+1)^3 - 7(k+1) + 3$$
$$= k^3 + 1 + 3k(k+1) - 7k - 7 + 3$$
$$= k^3 - 7k + 3 + 3k(k+1) - 6$$
$$= 3q + 3[k(k+1) - 2]$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true. [from step II]

So, by the principle of mathematical induction $P(n)$: is true for all natural number n.

Q. 6 $3^{2n} - 1$ **is divisible by 8, for all natural numbers** n.

Sol. Let $P(n): 3^{2n} - 1$ is divisible by 8, for all natural numbers.

Step **I** We observe that $P(1)$ is true.

$$P(1): 3^{2(1)} - 1 = 3^2 - 1$$
$$= 9 - 1 = 8, \text{which is divisible by 8.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): 3^{2k} - 1 = 8q$$

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): 3^{2(k+1)} - 1$$
$$= 3^{2k} \cdot 3^2 - 1$$
$$= 3^{2k} \cdot (8+1) - 1$$
$$= 8 \cdot 3^{2k} + 3^{2k} - 1$$
$$= 8 \cdot 3^{2k} + 8q$$
$$= 8(3^{2k} + q) \quad \text{[from step II]}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all natural numbers n.

Q. 7 For any natural numbers n, $7^n - 2^n$ **is divisible by 5.**

Sol. Consider the given statement is

$P(n): 7^n - 2^n$ is divisible by 5, for any natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1) = 7^1 - 2^1 = 5, \text{ which is divisible by 5.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = 7^k - 2^k = 5q$$

Step **III** Now, to prove $P(k+1)$ is true,

$$P(k+1): 7^{k+1} - 2^{k+1}.$$
$$= 7^k \cdot 7 - 2^k \cdot 2$$

$$= 7^k \cdot (5+2) - 2^k \cdot 2$$
$$= 7^k \cdot 5 + 2 \cdot 7^k - 2^k \cdot 2$$
$$= 5 \cdot 7^k + 2(7^k - 2^k)$$
$$= 5 \cdot 7^k + 2(5q)$$
$$= 5(7^k + 2q), \text{ which is divisible by 5.} \qquad \text{[from step II]}$$

So, $P(k+1)$ is true whenever $P(k)$ is true.
Hence, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Q. 8 For any natural numbers n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Sol. Let $P(n): x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
Step **I** We observe that $P(1)$ is true.
$$P(1): x^1 - y^1 = x - y$$
Step **II** Now, assume that $P(n)$ is true for $n = k$.
$$P(k): x^k - y^k \text{ is divisible by } (x - y).$$
$$\therefore \qquad x^k - y^k = q(x - y)$$
Step **III** Now, to prove $P(k+1)$ is true.
$$P(k+1): x^{k+1} - y^{k+1}$$
$$= x^k \cdot x - y^k \cdot y$$
$$= x^k \cdot x - x^k \cdot y + x^k \cdot y - y^k \cdot y$$
$$= x^k(x - y) + y(x^k - y^k)$$
$$= x^k(x - y) + yq(x - y)$$
$$= (x - y)[x^k + yq], \text{ which is divisible by } (x - y). \qquad \text{[from step II]}$$
Hence, $P(k+1)$ is true whenever $P(k)$ is true. So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Q. 9 $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Thinking Process

In step I put $n = 2$, the obtained result should be divisible by 6. Then, follow the same process as in question no. 4.

Sol. Let $P(n): n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
Step **I** We observe that $P(2)$ is true. $P(2): (2)^3 - 2$
$$\Rightarrow \qquad 8 - 2 = 6, \text{ which is divisible by 6.}$$
Step **II** Now, assume that $P(n)$ is true for $n = k$.
$$P(k): k^3 - k \text{ is divisible by 6.}$$
$$\therefore \qquad k^3 - k = 6q$$
Step **III** To prove $P(k+1)$ is true
$$P(k+1): (k+1)^3 - (k+1).$$
$$= k^3 + 1 + 3k(k+1) - (k+1)$$
$$= k^3 + 1 + 3k^2 + 3k - k - 1$$
$$= k^3 - k + 3k^2 + 3k$$
$$= 6q + 3k(k+1) \qquad \text{[from step II]}$$
We know that, $3k(k+1)$ is divisible by 6 for each natural number $n = k$.
So, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 10 $n(n^2 + 5)$ **is divisible by 6, for each natural number** n**.**

Sol. Let $P(n) : n(n^2 + 5)$ is divisible by 6, for each natural number n.

Step **I** We observe that $P(1)$ is true.

$$P(1) : 1(1^2 + 5) = 6, \text{ which is divisible by 6.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : k(k^2 + 5) \text{ is divisible by 6.}$$

$$\therefore \quad k(k^2 + 5) = 6q$$

Step **III** Now, to prove $P(k + 1)$ is true, we have

$$\begin{aligned} P(k+1) : (k+1)[(k+1)^2 + 5] &\\ &= (k+1)[k^2 + 2k + 1 + 5] \\ &= (k+1)[k^2 + 2k + 6] \\ &= k^3 + 2k^2 + 6k + k^2 + 2k + 6 \\ &= k^3 + 3k^2 + 8k + 6 \\ &= k^3 + 5k + 3k^2 + 3k + 6 \\ &= k(k^2 + 5) + 3(k^2 + k + 2) \\ &= (6q) + 3(k^2 + k + 2) \end{aligned}$$

We know that, $k^2 + k + 2$ is divisible by 2, where, k is even or odd.

Since, $P(k + 1) : 6q + 3(k^2 + k + 2)$ is divisible by 6. So, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true.

Q. 11 $n^2 < 2^n$**, for all natural numbers** $n \geq 5$**.**

Sol. Consider the given statement

$P(n) : n^2 < 2^n$ for all natural numbers $n \geq 5$.

Step **I** We observe that $P(5)$ is true

$$P(5) : 5^2 < 2^5$$

$$= 25 < 32$$

Hence, $P(5)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) = k^2 < 2^k \text{ is true.}$$

Step **III** Now, to prove $P(k + 1)$ is true, we have to show that

$$P(k+1) : (k+1)^2 < 2^{k+1}$$

Now, $\quad k^2 < 2^k = k^2 + 2k + 1 < 2^k + 2k + 1$

$$= (k+1)^2 < 2^k + 2k + 1 \quad \ldots \text{(i)}$$

Now, $(2k + 1) < 2^k \quad = 2^k + 2k + 1 < 2^k + 2^k$

$$= 2^k + 2k + 1 < 2 \cdot 2^k$$

$$= 2^k + 2k + 1 < 2^{k+1} \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get $(k + 1)^2 < 2^{k+1}$

So, $P(k + 1)$ is true, whenever $P(k)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all natural numbers $n \geq 5$.

Q. 12 $2n < (n + 2)!$ for all natural numbers n.

Sol. Consider the statement

$P(n): 2n < (n+2)!$ for all natural number n.

Step **I** We observe that, $P(1)$ is true. $P(1): 2(1) < (1+2)!$

$\Rightarrow \quad 2 < 3! \Rightarrow 2 < 3 \times 2 \times 1 \Rightarrow 2 < 6$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$,

$$P(k): 2k < (k+2)! \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true, we have to show that

$$P(k+1): 2(k+1) < (k+1+2)!$$

Now, $$2k < (k+2)!$$

$$2k + 2 < (k+2)! + 2$$

$$2(k+1) < (k+2)! + 2 \quad \ldots\text{(i)}$$

Also, $$(k+2)! + 2 < (k+3)! \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$2(k+1) < (k+1+2)!$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true.

Q. 13 $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Sol. Consider the statement

$P(n): \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Step **I** We observe that $P(2)$ is true.

$$P(2): \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true, we have to show that

$$P(k+1): \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k+1}} \text{ is true.}$$

Given that, $$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}}$$

$$\Rightarrow \quad \sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \quad \frac{(\sqrt{k})(\sqrt{k+1}) + 1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \quad \ldots\text{(i)}$$

If $$\sqrt{k+1} < \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$\Rightarrow \quad k + 1 < \sqrt{k}\,\sqrt{k+1} + 1$$

$$\Rightarrow \quad k < \sqrt{k(k+1)} \Rightarrow \sqrt{k} < \sqrt{k+1} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii),

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k+1}}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 14 $2 + 4 + 6 + \ldots + 2n = n^2 + n$, for all natural numbers n.

Sol. Let $P(n): 2 + 4 + 6 + \ldots + 2n = n^2 + n$

For all natural numbers n.

Step **I** We observe that $P(1)$ is true.

$$P(1): 2 = 1^2 + 1$$
$$2 = 2, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$\therefore$ $$P(k): 2 + 4 + 6 + \ldots + 2k = k^2 + k$$

Step **III** To prove that $P(k+1)$ is true.

$$P(k+1): 2 + 4 + 6 + 8 + \ldots + 2k + 2(k+1)$$
$$= k^2 + k + 2(k+1)$$
$$= k^2 + k + 2k + 2$$
$$= k^2 + 2k + 1 + k + 1$$
$$= (k+1)^2 + k + 1$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, $P(n)$ is true.

Q. 15 $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$ for all natural numbers n.

Sol. Consider the given statement

$$P(n): 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1, \text{ for all natural numbers } n$$

Step **I** We observe that $P(0)$ is true.

$$P(1): 1 = 2^{0+1} - 1$$
$$1 = 2^1 - 1$$
$$1 = 2 - 1$$
$$1 = 1, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

So, $P(k): 1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$ is true.

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): 1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1}$$
$$= 2^{k+1} - 1 + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$
$$= 2^{(k+1)+1} - 1$$

So, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, $P(n)$ is true.

Q. 16 $1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$, for all natural numbers n.

Sol. Let $P(n): 1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$, for all natural numbers n.

Step **I** We observe that $P(1)$ is true.

$$P(1): 1 = 1(2 \times 1 - 1),\ 1 = 2 - 1 \text{ and } 1 = 1, \text{ which is true.}$$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

So, $P(k): 1 + 5 + 9 + \ldots + (4k - 3) = k(2k - 1)$ is true.

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): 1+5+9+\dots+(4k-3)+4(k+1)-3$$
$$= k(2k-1)+4(k+1)-3$$
$$= 2k^2 - k + 4k + 4 - 3$$
$$= 2k^2 + 3k + 1$$
$$= 2k^2 + 2k + k + 1$$
$$= 2k(k+1) + 1(k+1)$$
$$= (k+1)(2k+1)$$
$$= (k+1)[2k+1+1-1]$$
$$= (k+1)[2(k+1)-1]$$

So, $P(k+1)$ is true, whenever $p(k)$ is true, hence $P(n)$ is true.

Long Answer Type Questions

Use the Principle of Mathematical Induction in the following questions.

Q. 17 A sequence $a_1, a_2, a_3, \dots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Sol. A sequence $a_1, a_2, a_3, \dots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$, for all natural numbers $k \geq 2$.

Let $P(n): a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Step **I** We observe $P(2)$ is true.

For $n = 2$, $a_2 = 3 \cdot 7^{2-1} = 3 \cdot 7^1 = 21$ is true.

As $a_1 = 3,\ a_k = 7a_{k-1}$

$\Rightarrow$ $a_2 = 7 \cdot a_{2-1} = 7 \cdot a_1$

$\Rightarrow$ $a_2 = 7 \times 3 = 21$ $[\because a_1 = 3]$

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k): a_k = 3 \cdot 7^{k-1}$$

Step **III** Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1): a_{k+1} = 3 \cdot 7^{k+1-1}$$

$$a_{k+1} = 7 \cdot a_{k+1-1} = 7 \cdot a_k$$
$$= 7 \cdot 3 \cdot 7^{k-1} = 3 \cdot 7^{k-1+1}$$

So, $P(k+1)$ is true, whenever $p(k)$ is true. Hence, $P(n)$ is true.

Q. 18 A sequence $b_0, b_1, b_2, \dots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k. Show that $b_n = 5 + 4n$, for all natural number n using mathematical induction.

Sol. Consider the given statement,

$P(n): b_n = 5 + 4n$, for all natural numbers given that $b_0 = 5$ and $b_k = 4 + b_{k-1}$

Step **I** $P(1)$ is true.

$$P(1): b_1 = 5 + 4 \times 1 = 9$$

As $b_0 = 5, b_1 = 4 + b_0 = 4 + 5 = 9$

Hence, $P(1)$ is true.

Step **II** Now, assume that $P(n)$ is true for $n = k$.

$$P(k) : b_k = 5 + 4k$$

Step **III** Now, to prove $P(k + 1)$ is true, we have to show that

$$\therefore \quad P(k+1) : b_{k+1} = 5 + 4(k + 1)$$

$$b_{k+1} = 4 + b_{k+1-1}$$
$$= 4 + b_k$$
$$= 4 + 5 + 4k = 5 + 4(k + 1)$$

So, by the mathematical induction $P(k + 1)$ is true whenever $P(k)$ is true, hence $P(n)$ is true.

Q. 19 A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$, for all natural numbers, $k \geq 2$. Show that $d_n = \dfrac{2}{n!}$, for all $n \in N$.

Sol. Let $P(n) : d_n = \dfrac{2}{n!}, \forall n \in N$, to prove $P(2)$ is true.

Step **I** $$P(2) : d_2 = \frac{2}{2!} = \frac{2}{2 \times 1} = 1$$

As, given $$d_1 = 2$$

$$\Rightarrow \quad d_k = \frac{d_{k-1}}{k}$$

$$\Rightarrow \quad d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$$

Hence, $P(2)$ is true.

Step **II** Now, assume that $P(k)$ is true.

$$P(k) : d_k = \frac{2}{k!}$$

Step **III** Now, to prove that $P(k + 1)$ is true, we have to show that $P(k+1) : d_{k+1} = \dfrac{2}{(k+1)!}$

$$d_{k+1} = \frac{d_{k+1-1}}{k} = \frac{d_k}{k}$$
$$= \frac{2}{k!k} = \frac{2}{(k+1)!}$$

So, $P(k + 1)$ is true. Hence, $P(n)$ is true.

Q. 20 Prove that for all $n \in N$

$$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$$
$$= \frac{\cos\left[\alpha + \left(\dfrac{n-1}{2}\right)\beta\right]\sin\left(\dfrac{n\beta}{2}\right)}{\sin\dfrac{\beta}{2}}$$

Thinking Process

To prove this, use the formula $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ *and*

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right)$$

Sol. Let $P(n): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(n-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Step I We observe that $P(1)$

$$P(1): \cos\alpha = \frac{\cos\left[\alpha + \left(\frac{1-1}{2}\right)\right]\beta\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \frac{\cos(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\cos\alpha = \cos\alpha$$

Hence, $P(1)$ is true.

Step II Now, assume that $P(n)$ is true for $n = k$.

$$P(k): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\right]\beta\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}}$$

Step III Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1): \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$$

$$+ \cos[\alpha+(k+1-1)\beta] = \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin(k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$\text{LHS} = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta] + \cos(\alpha+k\beta)$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} + \cos(\alpha+k\beta)$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2} + \cos(\alpha+k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}+\frac{k\beta}{2}\right) - \sin\left(\alpha+\frac{k\beta}{2}-\frac{\beta}{2}-\frac{k\beta}{2}\right) + \sin\left(\alpha+k\beta+\frac{\beta}{2}\right) - \sin\left(\alpha+k\beta-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha+k\beta+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\frac{1}{2}\left(\alpha+\frac{\beta}{2}+k\beta+\alpha-\frac{\beta}{2}\right)\sin\frac{1}{2}\left(\alpha+\frac{\beta}{2}+k\beta-\alpha+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\frac{2\alpha+k\beta}{2}\right)\sin\left(\frac{k\beta+\beta}{2}\right)}{\sin\frac{\beta}{2}} = \frac{\cos\left(\alpha+\frac{k\beta}{2}\right)\sin(k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \text{RHS}$$

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 21 Prove that $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}, \forall n \in N.$

Sol. Let $P(n): \cos\theta \cos 2\theta \ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

Step **I** For $n = 1$, $P(1): \cos\theta = \dfrac{\sin 2^1\theta}{2^1 \sin\theta}$

$$= \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$$

which is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$$P(k): \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ldots \cos 2^{k-1}\theta = \frac{\sin 2^k\theta}{2^k \sin\theta} \text{ is true.}$$

Step **III** To prove $P(k+1)$ is true.

$$P(k+1): \cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ldots \cos 2^{k-1}\theta \cdot \cos 2^k\theta$$

$$= \frac{\sin 2^k\theta}{2^k \sin\theta} \cdot \cos 2^k\theta$$

$$= \frac{2\sin 2^k\theta \cdot \cos 2^k\theta}{2 \cdot 2^k \sin\theta}$$

$$= \frac{\sin 2 \cdot 2^k\theta}{2^{k+1}\sin\theta} = \frac{\sin 2^{(k+1)}\theta}{2^{k+1}\sin\theta}$$

which is true.

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Q. 22 Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin\dfrac{n\theta}{2} \sin\dfrac{(n+1)}{2}\theta}{\sin\dfrac{\theta}{2}}$, for all $n \in N$.

Thinking Process

To use the formula of $2\sin A \sin B = \cos(A - B) - \cos(A + B)$ *and*

$\cos A - \cos B = 2\sin\dfrac{A+B}{2} \cdot \sin\dfrac{B-A}{2}$ *also* $\cos(-\theta) = \cos\theta$.

Sol. Consider the given statement

$P(n): \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\dfrac{n\theta}{2} \sin\dfrac{(n+1)\theta}{2}}{\sin\dfrac{\theta}{2}}, \text{ for all } n \in N$$

Step **I** We observe that $P(1)$ is

$$P(1): \sin\theta = \frac{\sin\dfrac{\theta}{2} \cdot \sin\dfrac{(1+1)}{2}\theta}{\sin\dfrac{\theta}{2}} = \frac{\sin\dfrac{\theta}{2} \cdot \sin\theta}{\sin\dfrac{\theta}{2}}$$

$$\sin\theta = \sin\theta$$

Hence, $P(1)$ is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$$P(k): \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta$$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} \text{ is true.}$$

Step **III** Now, to prove $P(k+1)$ is true.

$$P(k+1): \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta + \sin(k+1)\theta$$

$$= \frac{\sin\frac{(k+1)\theta}{2}\sin\left(\frac{k+1+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$\text{LHS} = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta + \sin(k+1)\theta$$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} + \sin(k+1)\theta = \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta + \sin(k+1)\theta\cdot\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left[\frac{k\theta}{2} - \left(\frac{k+1}{2}\right)\theta\right] - \cos\left[\frac{k\theta}{2} + \left(\frac{k+1}{2}\right)\theta\right] + \cos\left[(k+1)\theta - \frac{\theta}{2}\right] - \cos\left[(k+1)\theta + \frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{\theta}{2}\right) + \cos\left(k\theta + \frac{\theta}{2}\right) - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}} = \frac{2\sin\frac{1}{2}\left(\frac{\theta}{2} + k\theta + \frac{3\theta}{2}\right)\cdot\sin\frac{1}{2}\left(k\theta + \frac{3\theta}{2} - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{k\theta + 2\theta}{2}\right)\cdot\sin\left(\frac{k\theta + \theta}{2}\right)}{\sin\frac{\theta}{2}} = \frac{\sin(k+1)\frac{\theta}{2}\cdot\sin(k+1+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 23 Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

Thinking Process

Here, use the formula $(a+b)^5 = a^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + b^5$

and $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

Sol. Consider the given statement

$P(n): \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

Step **I** We observe that $P(1)$ is true.

$P(1): \frac{(1)^5}{5} + \frac{1^3}{3} + \frac{7(1)}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1$, which is a natural number. Hence, $P(1)$ is true.

Step **II** Assume that $P(n)$ is true, for $n = k$.

$P(k): \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is natural number.

Step **III** Now, to prove $P(k+1)$ is true.

$$\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$$
$$=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+1+3k(k+1)}{3}+\frac{7k+7}{15}$$
$$=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+1+3k^2+3k}{3}+\frac{7k+7}{15}$$
$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+\frac{5k^4+10k^3+10k^2+5k+1}{5}+\frac{3k^2+3k+1}{3}+\frac{7k+7}{15}$$
$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+2k^2+k+k^2+k+\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$$
$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+3k^2+2k+1, \text{ which is a natural number}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 24 Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

Sol. Consider the given statement

$P(n): \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

Step **I** We observe that, $P(2)$ is true,

$$P(2): \frac{1}{2+1}+\frac{1}{2+2}>\frac{13}{24}.$$
$$\frac{1}{3}+\frac{1}{4}>\frac{13}{24}$$
$$\frac{4+3}{12}>\frac{13}{24}$$
$$\frac{7}{12}>\frac{13}{24}, \text{ which is true.}$$

Hence, $P(2)$ is true.

Step **II** Now, we assume that $P(n)$ is true,

For $n=k$,

$$P(k): \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}>\frac{13}{24}.$$

Step **III** Now, to prove $P(k+1)$ is true, we have to show that

$$P(k+1): \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$$

Given, $$\frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}>\frac{13}{24}$$
$$\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}+\frac{1}{2(k+1)}$$
$$\frac{13}{24}+\frac{1}{2(k+1)}>\frac{13}{24}$$
$$\because \quad \frac{1}{k+1}+\frac{1}{k+2}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$$

So, $P(k+1)$ is true, whenever $P(k)$ is true. Hence, $P(n)$ is true.

Q. 25 Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in N$.

Sol. Let $P(n)$: Number of subset of a set containing n distinct elements is 2^n, for all $n \in N$.

Step **I** We observe that $P(1)$ is true, for $n = 1$.

Number of subsets of a set contain 1 element is $2^1 = 2$, which is true.

Step **II** Assume that $P(n)$ is true for $n = k$.

$P(k)$: Number of subsets of a set containing k distinct elements is 2^k, which is true.

Step **III** To prove $P(k+1)$ is true, we have to show that

$P(k+1)$: Number of subsets of a set containing $(k+1)$ distinct elements is 2^{k+1}.

We know that, with the addition of one element in the set, the number of subsets become double.

$\therefore$ Number of subsets of a set containing $(k+1)$ distinct elements $= 2 \times 2^k = 2^{k+1}$.

So, $P(k+1)$ is true. Hence, $P(n)$ is true.

Objective Type Questions

Q. 26 If $10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$, then the least positive integral value of k is

(a) 5 (b) 3 (c) 7 (d) 1

Sol. *(a)* Let $P(n): 10^n + 3 \cdot 4^{n+2} + k$ is divisible by 9, for all $n \in N$.

For $n = 1$, the given statement is also true $10^1 + 3 \cdot 4^{1+2} + k$ is divisible by 9.

$$\because \quad = 10 + 3 \cdot 64 + k = 10 + 192 + k$$
$$= 202 + k$$

If $(202 + k)$ is divisible by 9, then the least value of k must be 5.

$$\because \quad 202 + 5 = 207 \text{ is divisible by } 9$$
$$\Rightarrow \quad \frac{207}{9} = 23$$

Hence, the least value of k is 5.

Q. 27 For all $n \in N$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by

(a) 19 (b) 17 (c) 23 (d) 25

Sol. *(b, c)*

Given that, $3 \cdot 5^{2n+1} + 2^{3n+1}$

For $n = 1$,

$$3 \cdot 5^{2(1)+1} + 2^{3(1)+1}$$
$$= 3 \cdot 5^3 + 2^4$$
$$= 3 \times 125 + 16 = 375 + 16 = 391$$

Now, $\quad 391 = 17 \times 23$

which is divisible by both 17 and 23.

Q. 28 If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Let $P(n) : x^n - 1$ is divisible by $(x - k)$.

For $n = 1$, $x^1 - 1$ is divisible by $(x - k)$.

Since, if $x - 1$ is divisible by $x - k$. Then, the least possible integral value of k is 1.

Fillers

Q. 29 If $P(n) : 2n < n!$, $n \in N$, then $P(n)$ is true for all $n \geq$ ....

Sol. Given that, $P(n) : 2n < n!$, $n \in N$

For $n = 1$, $2 < !$ [false]

For $n = 2$, $2 \times 2 < 2!4 < 2$ [false]

For $n = 3$, $2 \times 3 < 3!$

$6 < 3!$

$6 < 3 \times 2 \times 1$

$(6 < 6)$ [false]

For $n = 4$, $2 \times 4 < 4!$

$8 < 4 \times 3 \times 2 \times 1$

$(8 < 24)$ [true]

For $n = 5$, $2 \times 5 < 5!$

$10 < 5 \times 4 \times 3 \times 2 \times 1$

$(10 < 120)$ [true]

Hence, $P(n)$ is for all $n \geq 4$.

True/False

Q. 30 Let $P(n)$ be a statement and let $P(k) \Rightarrow P(k + 1)$, for some natural number k, then $P(n)$ is true for all $n \in N$.

Sol. ***False***

The given statement is false because $P(1)$ is true has not been proved.